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Heavy- to light-meson transition form factors

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Abstract

Semileptonic heavy → heavy and heavy → light meson transitions are studied as a phenomenological application of a heavy-quark limit of Dyson-Schwinger equations. Employing two parameters: E, the difference between the mass of the heavy meson and the effective-mass of the heavy quark; and Λ, the width of the heavy-meson Bethe-Salpeter amplitude, we calculate $f_+(t)$ for all decays on their entire kinematically accessible t-domain. Our study favours f_B in the range 0.135-0.17 GeV and with $E = 0.44\,$GeV and $1/\Lambda = 0.14\,$fm we obtain $f_+^{B\pi}(0) = 0.46$. As a result of neglecting $1/m_c$-corrections, we estimate that our calculated values of $\rho^2 = 0.87$ and $f_+^{DK}(0) = 0.62$ are too low by approximately 15%. However, the bulk of these corrections should cancel in our calculated values of $Br(D \to \pi\ell\nu)/Br(D \to K\ell\nu) = 0.13$ and $f_+^{D\pi}(0)/f_+^{DK}(0) = 1.16$.

I. INTRODUCTION

Semileptonic meson decays are simple, experimentally accessible and only have a single hadron in the initial and final state. They are flavour-changing weak interaction processes and hence can be used as a means of extracting the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, which characterise the difference between the mass eigenstates and the weak eigenstates in the Standard Model. For example, the $K^+ \to \pi^0 e^+ \nu_e$ (K^+_{e3}) and $K^0_L \to \pi^\pm e^\mp \nu_e$ (K^0_{e3}) decays currently provide the most accurate determination of $|V_{us}|(= 0.2196 \pm 0.0023)$ [1], which would have been $\sin\theta_c$ in the Cabibbo theory of weak interactions. The mechanism of the weak interaction is well understood. Hence, like elastic, electromagnetic form factors, these decays can also be used as a tool to probe the structure of the hadrons in the initial and final state.

A major goal of current B-meson experiments is to determine accurately the matrix elements V_{cb} and V_{ub}, the first of which is accessible via $B \to D(D^*)\ell\nu$ decays and the latter via $B \to \pi(\rho)\ell\nu$. The decays with a pseudoscalar meson in both the initial and final state are the simplest to study theoretically because they are only sensitive to the vector coupling of the W-boson to the quarks and only two form factors are needed for a complete description. However, experimentally those with a vector meson in the final state provide the best statistics because the decay can proceed via both S- and D-waves.

The $B \to D(D^*)\ell\nu$ decays proceed via a $b \to c$ transition and experimentally this is the closest one can come to realising a "heavy $\to$ heavy" transition [2]. It is in the analysis of these decays that heavy-quark symmetry [3]; i.e., an expansion of observables in $\Lambda_{\rm QCD}/m_f$, where m_f is the current-quark mass of the $f = b, c$ quark, is most likely to be of use. However, in reality the $\Lambda_{\rm QCD}/m_c$-corrections, in particular, may nevertheless be large ($\sim$ 30)% and difficult to estimate in this case.

Heavy Quark Effective Theory (HQET) [3] provides a systematic method for exploring the consequences of heavy-quark symmetry. It can be used to reduce the number of independent form factors required to describe heavy $\to$ heavy decays, relating them to a minimal number of so-called "universal" form factors. However, it can't be used to calculate the q^2-dependence of the form factors. This depends on the internal structure of the heavy mesons and its calculation requires the application of nonperturbative techniques. One such technique and its application to the calculation of these form factors is our focus herein.

The methods of HQET are also not directly applicable to the decays $B \to \pi(\rho)\ell\nu_\ell$, $D \to K\ell\nu_\ell$ and $D \to \pi\ell\nu_\ell$, all of which have light mesons in the final state. A primary impediment is that the current-quark mass of the s-quark $m_s \sim O(\Lambda_{\rm QCD})$, hence $\Lambda_{\rm QCD}/m_s$ is not a suitable expansion parameter. In addition, a theoretical description of these decays requires a good understanding of light quark propagation characteristics and the internal structure of light mesons. This is provided by the extensive body of Dyson-Schwinger equation (DSE) studies [4,5] in QCD.

The DSEs are a system of coupled integral equations whose solutions, the n-point Schwinger functions, are the fully-dressed Euclidean propagators and vertices for the theory. Once all the Schwinger functions are known then the theory is completely specified. To arrive at a tractable problem one must truncate the system at a given level. Truncations that preserve the global symmetries of a field theory are easy to implement [6]. Preserving the gauge symmetry is more difficult but progress is being made [7].

In a general covariant gauge the dressed-gluon 2-point Schwinger function (Euclidean propagator), $D_{\mu\nu}(k)$, is characterised by a single scalar function, which we denote $\mathcal{G}(k^2)/k^2$. Important here is the particular, qualitatively robust result of studies of the DSE for $D_{\mu\nu}(k)$ that $\mathcal{G}(k^2)/k^2$ is strongly enhanced in the infrared; i.e, its behaviour in the vicinity of $k^2=0$ can be represented as a distribution [8,9]. The infrared enhancement in $D_{\mu\nu}(k)$ becomes prominent for $k^2 \sim 1\,\mathrm{GeV}^2$ and is not peculiar to covariant gauges [10].

The dressed-quark propagator for a quark of flavour f can be written in the general form[1]

$$S_f(p) = \frac{Z_f(p^2)}{i\gamma\cdot p + M_f(p^2)}\,, \tag{1}$$

where $Z_f(p^2)$ is the momentum-dependent wavefunction renormalisation and $M_f(p^2)$ is the momentum-dependent quark mass-function. The dressed-gluon propagator is an important element in the kernel of the DSE satisfied by $S_f(p)$. In existing studies of this DSE that employ a dressed-quark-gluon vertex that is free of kinematic, light-cone singularities, the infrared enhancement in $D_{\mu\nu}(k)$ is sufficient to ensure that $S(p)$ doesn't have a Lehmann representation. This entails the absence of coloured quark states from the spectrum; i.e., quark confinement [11]. If $\mathcal{G}(k^2)<\infty$ at $k^2=0$ it is possible to obtain a solution, $S_f(p)$, of the quark DSE that has a Lehmann representation [12].

There is another important consequence of the infrared enhancement in $\mathcal{G}(k^2)/k^2$. The enhancement is characterised by a mass-scale $\omega \sim \Lambda_{\rm QCD}$ and for light quarks; i.e, u-, d- and s-quarks for which $m_f \leq \Lambda_{\rm QCD}$, it generates a significant enhancement in $M_f(p^2)$. A single, indicative and quantitative measure of this enhancement in $M_f(p^2)$ is the ratio M_f^E/m_f, where M_f^E is the Euclidean constituent-quark mass defined as the solution of $p^2 = M^2(p^2)$.[2] The results

$$\frac{M^E_{u,d}}{m_{u,d}} \sim 150\,,\;\; \frac{M^E_s}{m_s} \sim 10 \tag{2}$$

demonstrate that the infrared enhancement in $\mathcal{G}(k^2)/k^2$ leads to at least an order-of-magnitude infrared enhancement in $M_f(p^2)$. It is nonperturbative in origin[3] and has important qualitative and quantitative implications for light meson observables, as illustrated in Refs. [5,13].

The effect of the infrared enhancement in $\mathcal{G}(k^2)/k^2$ on $M_{c,b}(p^2)$ is much less dramatic [14]:

$$\frac{M^E_b}{m_b} \sim 1.5\,,\;\; \frac{M^E_c}{m_c} \sim 2.0\,. \tag{3}$$

[1]We employ a Euclidean metric with $\delta_{\mu\nu} = \mathrm{diag}(1,1,1,1)$, $\gamma_\mu^\dagger = \gamma_\mu$ and $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$. A spacelike 4-vector, k_μ, has $k^2>0$.

[2]Quark confinement entails that there is no "pole-mass" [11], which would be the solution of $p^2 + M^2(p^2) = 0$.

[3]The renormalisation-point-dependence of the current-quark mass affects the actual value of the ratio M_f^E/m_f but not the qualitative features of this discussion.

In this case $m_f \gg \Lambda_{\rm QCD}$ and the momentum-dependence of $M_{c,b}(p^2)$ is primarily perturbative in origin. As observed in Ref. [15] it is therefore a good approximation to write

$$M_b(p^2) = \text{ const} := \hat{M}_b \approx M_b^E \,, \tag{4}$$

for $p^2 \gg -m_b^2$, although the b-quark is still confined and there is no pole mass. For the same reason $Z_b(p^2) \equiv 1$ is also a good approximation. This and Eq. (4) form the basis of the heavy-quark limit of the DSEs explored in Ref. [15] wherein, on the domain explored by heavy $\to$ heavy semileptonic decays, the dressed-b-quark propagator was approximated by

$$S_b(p) = \frac{1}{i\gamma\cdot p + \hat{M}_b} \,. \tag{5}$$

In Ref. [15] the dressed-c-quark propagator was approximated by an analogous expression:

$$S_c(p) = \frac{1}{i\gamma\cdot p + \hat{M}_c} \,. \tag{6}$$

However, the justification of this is less certain because the momentum dependence of $Z_c(p^2)$ and $M_c(p^2)$ is significantly more rapid. The approach employed in Ref. [16] is one means of exploring the fidelity of this approximation, as are the direct studies for which Ref. [14] is the pilot.

Our aim herein is a unified description and correlation of semileptonic heavy $\to$ heavy and heavy $\to$ light meson transitions as an extension of the application of DSE methods. We follow Ref. [15] in describing the b- and c-quark propagators by Eqs. (5) and (6), respectively, and in our analysis we consider the effects and limitations of Eq. (6). These equations represent the primary, exploratory hypothesis in our study because the propagation characteristics of light-quarks and the structure of light meson bound states is well understood following the extensive application of DSE methods in this domain [5,13,17]. In Sec. II we define our approximation to the matrix elements describing $B(D) \to \pi(K)\ell\nu$ transitions and fully specify a heavy-quark limit of our DSE application. Our results are presented and discussed in Sec. III and we make some concluding remarks in Sec. IV.

II. SEMILEPTONIC DECAYS

Our primary focus is the pseudoscalar $\to$ pseudoscalar semileptonic decay

$$P_{H_1}(p_1) \to P_{H_2}(p_2)\,\ell\,\nu \,, \tag{7}$$

where P_{H_1} represents either a B or D meson with momentum p_1 ($p_1^2 = -m_{H_1}^2$) and P_{H_2} can be a D, K or π meson with momentum p_2 ($p_2^2 = -m_{H_2}^2$). The momentum transfer to the lepton pair is $q := p_1 - p_2$. A review of these decays is provided in Ref. [2] and a theoretical study of the light $\to$ light transitions is presented in Ref. [18].

The invariant amplitude describing the decay is

$$A(P_{H_1} \to P_{H_2}\ell\nu) = \frac{G_F}{\sqrt{2}}\, V_{qQ}\, \bar{\ell}\gamma_\mu(1-\gamma_5)\nu\, M_\mu^{P_{H_1}P_{H_2}}(p_1,p_2) \,, \tag{8}$$

where G_F is the Fermi weak-decay constant, V_{qQ} is the appropriate element of the CKM matrix and the hadronic current is

$$M_\mu^{P_{H_1} P_{H_2}}(p_1, p_2) := \langle P_{H_2}(p_2) | \bar{q} \gamma_\mu Q | P_{H_1}(p_1) \rangle \tag{9}$$
$$= f_+(t)(p_1 + p_2)_\mu + f_-(t) q_\mu \,, \tag{10}$$

with $t := -q^2$. The form factors, $f_\pm(t)$, contain all the information about strong interaction effects in these processes and their accurate estimation is essential to the extraction of V_{qQ} from a measurement of a semileptonic decay rate:

$$\Gamma(P_{H_1} \to P_{H_2} \ell \nu) = \frac{G_F^2}{192\pi^3} |V_{qQ}|^2 \frac{1}{m_{H_1}^3} \int_0^{t_-} dt \, |f_+(t)|^2 \left[(t_+ - t)(t_- - t)\right]^{3/2} \,, \tag{11}$$

with $t_\pm := (m_{H_1} \pm m_{H_2})^2$ and neglecting the lepton mass.

A. Impulse Approximation

In impulse approximation

$$M_\mu^{P_{H_1} P_{H_2}}(p_1, p_2) = \tag{12}$$
$$\frac{N_c}{16\pi^4} \int d^4k \, \mathrm{tr} \left[\bar{\Gamma}_{H_2}(k; -p_2) S_q(k + p_2) i \mathcal{V}_\mu^{qQ}(k + p_2, k + p_1) S_Q(k + p_1) \Gamma_{H_1}(k; p_1) S_{q'}(k) \right] \,,$$

where: $\Gamma_{H_1}(k; p_1)$ is the Bethe-Salpeter amplitude for the H_1 meson;

$$\bar{\Gamma}_{H_2}(k; -p_2)^t := C^\dagger \Gamma_{H_2}(-k; -p_2) C, \; C = \gamma_2 \gamma_4, \tag{13}$$

and M^t is the matrix transpose of M; and $\mathcal{V}_\mu^{qQ}(k_1, k_2)$ is the vector part of the dressed-quark-W-boson vertex.

1. Quark Propagators

The dressed quark propagators, $S_f(p)$, in Eq. (12) are the solution of

$$S(p)^{-1} = i\gamma \cdot p + m_{\mathrm{bm}} + \int \frac{d^4q}{(2\pi)^4} \, g^2 D_{\mu\nu}(p - q) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma_\nu^a(q, p) \,, \tag{14}$$

where: $D_{\mu\nu}(k)$ is the dressed-gluon propagator; $\Gamma_\nu^a(q, p)$ is the dressed-quark-gluon vertex; m_{bm} is the current-quark bare mass; and one can write $S_f(p)$ in the general form

$$S_f(p) = -i\gamma \cdot p \, \sigma_V^f(p^2) + \sigma_S^f(p^2) \,, \tag{15}$$

which is completely equivalent to Eq. (1). A thorough discussion of the numerical solution of Eq. (14), including a discussion of renormalisation, is given in Ref. [17].

a. Light quarks. Herein, for the light u-, d- and s-quark propagators, we do not directly employ a numerical solution of Eq. (14). Instead we use the algebraic parametrisations of these solutions developed in Ref. [19] because they efficiently characterise the essential and robust elements of the solution obtained in many studies [4] of the quark DSE:

$$\bar{\sigma}_S^f(x) = 2\bar{m}_f \mathcal{F}(2(x+\bar{m}_f^2)) + \mathcal{F}(b_1 x)\mathcal{F}(b_3 x)\left(b_0^f + b_2^f \mathcal{F}(\epsilon x)\right) , \tag{16}$$

$$\bar{\sigma}_V^f(x) = \frac{2(x+\bar{m}_f^2) - 1 + e^{-2(x+\bar{m}_f^2)}}{2(x+\bar{m}_f^2)^2} , \tag{17}$$

where: $f = u, s$ (isospin symmetry is assumed),

$$\mathcal{F}(y) := \frac{1-\mathrm{e}^{-y}}{y} ; \tag{18}$$

$x = p^2/(2D)$; $\bar{m}_f = m_f/\sqrt{2D}$; and

$$\bar{\sigma}_S^f(x) := \sqrt{2D}\, \sigma_S^f(p^2) , \tag{19}$$

$$\bar{\sigma}_V^f(x) := 2D\, \sigma_V^f(p^2) , \tag{20}$$

with D a mass scale. This algebraic form combines the effects of confinement and dynamical chiral symmetry breaking with free-particle (asymptotically-free) behaviour at large, spacelike p^2. The parameters $\bar{m}_f$, $b^f_{0\ldots3}$ in Eqs. (16) and (17) take the values

$$\begin{array}{c|ccccc} & \bar{m}_f & b_0^f & b_1^f & b_2^f & b_3^f \\ \hline u: & 0.00897 & 0.131 & 2.90 & 0.603 & 0.185 \\ s: & 0.224 & 0.105 & \underline{2.90} & 0.740 & \underline{0.185} \end{array} , \tag{21}$$

which were determined in a least-squares fit to a range of light-hadron observables. The values of $b^s_{1,3}$ are underlined to indicate that the constraints $b^s_{1,3} = b^u_{1,3}$ were imposed in the fitting. The scale parameter $D = 0.160\,\mathrm{GeV}^2$.

b. Heavy quarks. As described in Sec. I, and exploited in Ref. [15], the momentum-dependence of $Z_f(p^2)$ and $M_f(p^2)$ is much weaker for the heavy-quarks than it is for the light-quarks. This is illustrated for two different but related DSE-models in Refs. [14,15] and justifies Eq. (5) for the b-quark and the cautious, exploratory use of Eq. (6) for the c-quark.

These equations provide the origin of heavy-quark symmetry in the DSE framework. Its elucidation is completed by introducing the heavy-meson velocity, v_μ, via

$$p_{1\mu} := m_{H_1}\, v_\mu := (\hat{M}_{f_Q} + E)\, v_\mu , \tag{22}$$

where $v^2 = -1$ and $E > 0$ is the difference between the heavy-meson mass and the effective-mass of the heavy-quark, $\hat{M}_{f_Q}$. Equations (5) and (6) then yield

$$S_{f_Q}(k+p_1) = \tfrac{1}{2}\, \frac{1 - i\gamma\cdot v}{k\cdot v - E} + \mathrm{O}\left(\frac{|k|}{\hat{M}_{f_Q}}, \frac{E}{\hat{M}_{f_Q}}\right) . \tag{23}$$

Exact heavy-quark symmetry arises from completely neglecting the $1/\hat{M}_{f_Q}$ corrections in all applications. The mass of the b-quark may justify this as a quantitatively reliable approximation but in making the same truncation for the c-quark one may expect quantitatively important corrections.

2. Bethe-Salpeter amplitude

As discussed in Refs. [14,17], the meson Bethe-Salpeter amplitudes in Eq. (12) are the solution of the homogeneous Bethe-Salpeter equation:

$$[\Gamma_H(k;P)]_{tu} = \int \frac{d^4q}{(2\pi)^4} \, [\chi_H(q;P)]_{sr} \, K^{rs}_{tu}(q,k;P) \,, \tag{24}$$

where

$$\chi_H(q;P) \doteq S_Q(q+P)\Gamma_H(q;P)S_{q'}(q) \,, \tag{25}$$

S_f are the dressed-quark propagators, and $r,\ldots,u$ represent colour-, Dirac- and flavour-matrix indices. In Eq. (24) $K^{rs}_{tu}(q,k;P)$ is the fully-amputated quark-antiquark scattering kernel. $K^{rs}_{tu}(q,k;P)$ is a 4-point Schwinger function obtained as the sum of a countable infinity of skeleton diagrams. It is two-particle-irreducible, with respect to the quark-antiquark pair of lines, and does not contain quark-antiquark to single gauge-boson annihilation diagrams, such as would describe the leptonic decay of a pseudoscalar meson. The numerical studies of Ref. [17] employed a ladder-like approximation:

$$K^{rs}_{tu}(q,k;P) = -g^2 D_{\mu\nu}(k-q) \left(\gamma_\mu \frac{\lambda^a}{2}\right)_{tr} \left(\gamma_\nu \frac{\lambda^a}{2}\right)_{su} \,, \tag{26}$$

which is consistent with the impulse approximation for $M_\mu^{P_{H_1} P_{H_2}}(p_1,p_2)$ and is a quantitatively reliable truncation for light, pseudoscalar mesons because of cancellations, order-by-order, between higher order diagrams in the skeleton expansion for K [6]. Ref. [14] is a first step in exploring the application of the methods of Ref. [17] to mesons containing at least one heavy quark.

a. Heavy meson Bethe-Salpeter amplitudes. Herein we do not use a numerical solution of Eq. (24) for the heavy-meson Bethe-Salpeter amplitude because we judge that our present studies are inadequate. One limitation, for example, is that simple ladder-like truncations do not yield the Dirac equation when the mass of one of the fermions becomes infinite and that defect may also be manifest in our study. Postponing the detailed exploration of this and other questions we employ instead an Ansatz motivated by the studies of Ref. [20] and used efficaciously in Ref. [15]:

$$\Gamma_{H_{1f}}(k;p_1) = \gamma_5 \left(1 + \tfrac{1}{2} i\gamma\cdot v\right) \tfrac{1}{\mathcal{N}_{H_{1f}}} \, \varphi(k^2) \,, \tag{27}$$

where $\mathcal{N}_{H_{1f}}$ is the canonical Bethe-Salpeter normalisation constant. Using Eq. (23)

$$\mathcal{N}^2_{H_{1f}} = \frac{1}{m_{H_{1f}}} \frac{N_c}{32\pi^2} \int_0^\infty du \, \varphi(z)^2 \, \left(\sigma_S^f(z) + \sqrt{u}\, \sigma_V^f(z)\right) := \frac{1}{m_{H_{1f}} \kappa_f^2} \,, \tag{28}$$

where $z = u - 2E\sqrt{u}$ and f labels the light-quark flavour.

In a solution of the Bethe-Salpeter equation the form of $\varphi(k^2)$ is completely determined. However, here it characterises our Ansatz and as our primary form we choose

$$\varphi(k^2) = \exp\left(-k^2/\Lambda^2\right) \,, \tag{29}$$

where Λ is a free parameter. In studies of heavy $\to$ heavy transitions [15] we found that, as long as $\varphi(k^2)$ is a non-negative, non-increasing, convex up function of k^2, the results were insensitive to its detailed form. As we shall see below, through a comparison of the results obtained using Eq. (29) and those obtained with

$$\tilde{\varphi}(k^2) = \frac{\tilde{\Lambda}^2}{k^2 + \tilde{\Lambda}^2}\,, \tag{30}$$

the same is true herein. Qualitatively, a primary requirement for an understanding of all the processes we consider is simply that the heavy meson be represented by a function that describes it as a finite-size, composite object: $1/\Lambda$ is a rough measure of that size.

b. Light meson Bethe-Salpeter amplitudes. Just as for the light-quark DSE, there have been numerous studies [4,5] of light mesons using Eq. (24) and a thorough discussion of the numerical solution, including a discussion of renormalisation, is presented in Ref. [17]. The light, pseudoscalar meson Bethe-Salpeter amplitude has the general form

$$\Gamma_H(k;P) = \gamma_5 \Big[i E_H(k;P) + \gamma\cdot P F_H(k;P) + \gamma\cdot k\, k\cdot P\, G_H(k;P) + \sigma_{\mu\nu}\, k_\mu P_\nu\, H_H(k;P)\Big]\,. \tag{31}$$

Until recently it was assumed that in quantitative phenomenological applications one could neglect all but $E_H(k;P)$ in describing the light, pseudoscalar meson and this was the assumption of Ref. [19]. However, a systematic study of the quark DSE and meson Bethe-Salpeter equation [17] demonstrates that the other functions are both qualitatively and quantitatively important. A reanalysis of elastic form factors using all amplitudes and refitting the parameters characterising the quark propagators is therefore necessary. It is underway but incomplete [21].

Herein we use the parametrisation of the light meson Bethe-Salpeter amplitude determined in Ref. [19] and the results [17] that it is a good approximation to treat $E_H(k;P) = E_H(k^2)$ and $E_\pi(k^2) = E_K(k^2) := \mathcal{E}(k^2)$; i.e., we use

$$\Gamma_{H=\pi,K}(k^2) = i\gamma_5\, \mathcal{E}(k^2)\,, \tag{32}$$

$$\mathcal{E}(k^2) = \frac{\sqrt{2}}{f_H}\,\frac{C_0\, \mathrm{e}^{-k^2/[2D]} + \sigma_S(k^2)|_{m_f=0}}{\sigma_V(k^2)|_{m_f=0}}\,, \tag{33}$$

where the parameter $C_0 = 0.214\,\mathrm{GeV}$ was fixed in Ref. [19] and therein yields the experimental value $f_\pi = 0.131$. For the kaon $f_K = 0.196\,\mathrm{GeV}$.

In principle, neglecting the other amplitudes in Eq. (31) is flawed. However, the light quark propagators of Eqs. (16)-(21) were also fixed under this assumption and it is the *combination* of these parametrisations in Eq. (25) that appears in the calculation of hadronic observables and reproduces available data. Therefore, if practiced judiciously, neglecting the other amplitudes can still provide quantitatively reliable results. To illustrate this we note that a preliminary reanalysis of the electromagnetic pion form factor [21], using all the amplitudes in Eq. (31) and refitting the u-quark propagator parameters in Eq. (21), yields results that are qualitatively indistinguishable from those obtained in Ref. [19] for $q^2 \leq 20\,\mathrm{GeV}^2$. It is only for $q^2 > 20\,\mathrm{GeV}^2$ that the qualitative and quantitative importance of F_π and G_π becomes manifest: these are the dominant amplitudes at large q^2 and ensure that $q^2 F_\pi(q^2) = \text{const}$, up to $\ln q^2$ corrections.

3. Quark-W-boson vertex

$\mathcal{V}_\mu^{qQ}(k_1,k_2)$ in Eq. (12) satisfies a DSE that describes both the strong and electroweak dressing of the vector part of the quark-W-boson vertex. Solving this equation is a problem that can be addressed using the methods of Ref. [17]. However, we postpone this problem for the present and note instead that from this DSE one can derive a Ward-Takahashi identity

$$(k_1-k_2)_\mu\, i\mathcal{V}_\mu^{f_1f_2}(k_1,k_2) = S_{f_1}^{-1}(k_1) - S_{f_2}^{-1}(k_2) - (m_{f_1}-m_{f_2})\,\Gamma_I^{f_1f_2}(k_1,k_2)\,, \tag{34}$$

where $\Gamma_I^{f_1f_2}(k_1,k_2)$ is the scalar vertex, which in the absence of interactions is simply the diagonal unit matrix in Dirac space. This identity can be used to constrain the form of $\mathcal{V}_\mu^{f_1f_2}(k_1,k_2)$, as the QED analogue has been used to constrain the dressed-quark-photon vertex [22].

When f_1 and f_2 are both heavy quarks then the ability to neglect gluon dressing, as manifest in Eq. (5), entails

$$(m_{f_1}-m_{f_2})\,\Gamma_I^{f_1f_2}(k_1,k_2) \approx (\hat{M}_{f_1}-\hat{M}_{f_2})\,1_D\,. \tag{35}$$

This justifies the approximation, used efficaciously in Ref. [15],

$$\mathcal{V}_\mu^{f_1f_2}(k_1,k_2) = \gamma_\mu \tag{36}$$

thereby amplifying the simplifications accruing in the heavy-quark limit. As demonstrated in Ref. [18], even in the case where both quarks are light, improvements to Eq. (36) only become quantitatively significant ($\sim 10\%$) in the magnitude of $f_+(t)$ at the extreme kinematic limit: $t=t_-$. Hence we use Eq. (36) in all calculations described herein.

B. Semileptonic decays in the heavy-quark limit

1. $B_f \to D_f$

Using Eqs. (15), (23) and (27), we find [15] from Eqs. (10) and (12) that, at leading order in $1/m_H$ where m_H is the heavy-meson mass,

$$f_\pm(t) = \tfrac{1}{2}\,\frac{m_{D_f}\pm m_{B_f}}{\sqrt{m_{D_f}m_{B_f}}}\,\xi_f(w)\,, \tag{37}$$

$$\xi_f(w) = \kappa_f^2\,\frac{N_c}{32\pi^2}\int_0^1 d\tau\,\frac{1}{W}\int_0^\infty du\,\varphi(z_W)^2\left[\sigma_S^f(z_W)+\sqrt{\frac{u}{W}}\sigma_V^f(z_W)\right]\,, \tag{38}$$

with $W=1+2\tau(1-\tau)(w-1)$, $z_W=u-2E\sqrt{u/W}$ and[4]

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[4]The minimum physical value of w is $w_{\rm min}=1$, which corresponds to maximum momentum transfer with the final state meson at rest; the maximum value is $w_{\rm max}\simeq(m_{B_f}^2+m_{D_f}^2)/(2m_{B_f}m_{D_f})=1.6$, which corresponds to maximum recoil of the final state meson with the charged lepton at rest.

$$w = \frac{m_{B_f}^2 + m_{D_f}^2 - t}{2m_{B_f} m_{D_f}} = v_{B_f} \cdot v_{D_f} \, . \tag{39}$$

The canonical normalisation of the Bethe-Salpeter amplitude, Eq. (28), automatically ensures that

$$\xi_f(w = 1) = 1 \, . \tag{40}$$

Equation (38) is an example of a general result that, in the heavy-quark limit, the semileptonic $H_f \to H_f'$ decays of heavy mesons are described by a single, universal function: $\xi_f(w)$ [23].

2. Heavy → Light

Using Eqs. (15), (23) and (27), and following the method outlined in the appendix, we find from Eqs. (10) and (12)

$$f_+^{H_1 H_2}(t) = \kappa_{q'} \frac{\sqrt{2}}{f_{H_2}} \frac{N_c}{32\pi^2} \, F_{q'}(t; E, m_{H_1}, m_{H_2}) \, , \tag{41}$$

where

$$F_{q'}(t; E, m_{H_1}, m_{H_2}) = \frac{4}{\pi} \int_{-1}^{1} \frac{d\gamma}{\sqrt{1-\gamma^2}} \int_0^1 d\nu \int_0^\infty u^2 du \, \varphi(z_1) \, \mathcal{E}(z_1) \, W_{q'}(\gamma, \nu, u) \, , \tag{42}$$

with

$$\begin{aligned} W_{q'}(\gamma, \nu, u) = {} & 2\tau^2 \left[\sigma_S^u(z_1) \frac{d}{dz_2} \sigma_V^{q'}(z_2) - \sigma_V^u(z_1) \frac{d}{dz_2} \sigma_S^{q'}(z_2) \right] \\ & + \left(1 - \frac{u\,\nu}{m_{H_1}} \right) \sigma_S^u(z_1) \, \sigma_V^{q'}(z_2) \\ & + \frac{1}{m_{H_1}} \Bigg[\sigma_S^u(z_1) \, \sigma_S^{q'}(z_2) + u\,\nu \, \sigma_V^u(z_1) \, \sigma_S^{q'}(z_2) \\ & \quad + (z_1 + u\,\nu \, M_{H_1}) \, \sigma_V^u(z_1) \, \sigma_V^{q'}(z_2) - 2m_{H_2}^2 \, \tau^2 \, \sigma_V^u(z_1) \, \frac{d}{dz_2} \sigma_V^{q'}(z_2) \Bigg] \end{aligned} \tag{43}$$

and

$$\begin{aligned} z_1 &= u^2 - 2u\,\nu\,E \, , & (44) \\ z_2 &= u^2 - 2u\,\nu\,(E - X) - m_{H_2}^2 + 2i\, m_{H_2} \, \gamma \, u \sqrt{1-\nu^2} \, , & (45) \\ X &= (m_{H_1}/2) \, [1 + (m_{H_2}^2 - t)/m_{H_1}^2] \, , & (46) \\ \tau &= u \sqrt{1-\nu^2} \sqrt{1-\gamma^2} \, . & (47) \end{aligned}$$

We note that because we have assumed isospin symmetry σ^u also represents a d-quark and, to illustrate Eq. (41), the $B^0 \to \pi^- \ell^+ \nu_\ell$ decay is characterised by

$$f_+^{B\pi}(t) = \kappa_d \frac{\sqrt{2}}{f_\pi} \frac{N_c}{32\pi^2} \, F_d(t; E, m_B, m_\pi) \, . \tag{48}$$

C. Leptonic decays of a heavy meson

We are also interested in the leptonic decay of a heavy, pseudoscalar meson, which is described by the matrix element

$$
\begin{aligned}
&\langle 0|\bar{q}\gamma_\mu\gamma_5 Q|P_{H_1}(p)\rangle := \\
&f_{H_1} p_\mu = \frac{N_c}{(2\pi)^4}\int d^4k\,\mathrm{tr}\left[\gamma_5\gamma_\mu S_Q(k+p)\Gamma_{H_1}(k;p)S_q(k)\right] ,
\end{aligned} \tag{49}
$$

where f_{H_1} is a single, dimensioned constant whose value describes all strong interaction contributions to this weak decay. For light mesons it has been studied extensively [4,17] and with this normalisation $f_\pi = 0.131\,\mathrm{GeV}$. Using Eqs. (15), (23), (27) and (28) one obtains an expression for f_{H_1} valid in the heavy-quark limit [15]:

$$
f_{H_1} = \frac{\kappa_f}{\sqrt{m_{H_1}}}\frac{N_c}{8\pi^2}\int_0^\infty du\,(\sqrt{u}-E)\,\varphi(z)\,\left[\sigma_S^f(z)+\tfrac{1}{2}\sqrt{u}\,\sigma_V^f(z)\right] , \tag{50}
$$

where $z = u - 2E\sqrt{u}$. It follows that in the heavy-quark limit

$$
f_{H_f} \propto \frac{1}{\sqrt{m_{H_f}}} . \tag{51}
$$

This scaling law is counter to the trend observed in the light mesons, as highlighted in Ref. [14], where f_H increases at least up to current-quark masses three-times that of the s-quark. Contemporary estimates of f_D and f_B, such as those analysed in Ref. [24], suggest that Eq. (51) is also not obeyed by experimentally accessible heavy mesons. The determination of the current-quark mass at which the light meson trend is reversed, and that at which this heavy-quark scaling law is satisfied, is an interesting, open question.

III. RESULTS AND DISCUSSION

We have now defined all that is necessary for our calculation of the semileptonic heavy $\to$ heavy and heavy $\to$ light meson transition form factors and heavy-meson leptonic decay constants. We have two free parameters: the binding energy, E, introduced in Eq. (22) and the width, Λ, of the heavy meson Bethe-Salpeter amplitude, introduced in Eq. (29). The dressed light-quark propagators and light-meson Bethe-Salpeter amplitudes have been fixed completely in the application of this framework to the study of π- and K-meson properties.

Our primary goal is to determine whether, with these two parameters, a description and correlation of existing data is possible using the DSE framework. This was certainly true in our analysis of heavy $\to$ heavy transitions alone [15]. We found that the function $\xi(w)$ necessarily has significant curvature and that a linear fit on $1 \leq w \leq 1.6$ is inconsistent with our study. However, our calculated value of the slope parameter

$$
\rho^2 := -\left.\frac{d}{dw}\xi(w)\right|_{w=1} \tag{52}
$$

was too strongly influenced by the experimental fit to the $B \to D$ data for that study to provide an independent prediction of ρ^2.[5] Herein we eliminate this bias by excluding D-meson observables from our primary procedure for fitting E and Λ. This also facilitates an elucidation of where $1/\hat{M}_c$-corrections are important.

Our key results are presented in column one of Table I. In obtaining these results we varied E and Λ in order to obtain a best, weighted least-squares fit to the three available lattice data points [25] for $f_+^{B\pi}$ and the experimental value [26] for the $B^0 \to \pi^- \ell^+ \nu$ branching ratio. In doing this we constrained our study to yield $f_B = 0.17\,\text{GeV}$ from Eq. (50), which is the central value favoured in a recent analysis of lattice simulations [24], and used $m_B = 5.27\,\text{GeV}$. This fitting procedure assumes only that the b-quark is in the heavy-quark domain; i.e., that $1/\hat{M}_b$-corrections to the formulae we have derived herein are negligible. Our calculated form of $f_+^{B\pi}(t)$ is presented in Fig. 1. A good *interpolation* of our result is provided by

$$f_+^{B\pi}(t) = \frac{0.458}{1 - t/m_{\text{mon}}^2}\,,\ m_{\text{mon}} = 5.67\,\text{GeV}\,. \tag{53}$$

This value of m_{mon} can be compared with that obtained in a fit to lattice data [25]: $m_{\text{mon}} = 5.6 \pm 0.3$.

In Table II we compare our favoured, calculated value of $f_+^{B\pi}(0) = 0.46$ with this quantity obtained using a range of other theoretical tools. Since the t-dependence of $f_+^{B\pi}(t)$ is an outcome of our calculation, the value we predict for $f_+^{B\pi}(0)$ is the only one that allows simultaneous agreement between our calculations and existing results of lattice simulations and the measured branching ratio. If these data are correct then in our framework it is not possible to obtain a value of $f_+^{B\pi}(0)$ that differs from this favoured value by more than 10% unless the calculated t-dependence is changed significantly. This could only be effected by a modification of the vertex Ansatz, Eq. (36), and hence the accuracy of our prediction can be seen as a test of the veracity of this Ansatz in the heavy-quark limit.

In Fig. 2 we present our calculated form of the function, $\xi(w)$, that characterises the semileptonic heavy $\to$ heavy meson decays. We have compared our calculation with the experimental results of Ref. [28] and the following fits to the experimental data in Ref. [29]:

$$\xi(w) = 1 - \rho^2\,(w-1),\ \rho^2 = 0.91 \pm 0.15 \pm 0.16\,, \tag{54}$$

$$\xi(w) = \frac{2}{w+1}\,\exp\left[(1-2\rho^2)\frac{w-1}{w+1}\right],\ \rho^2 = 1.53 \pm 0.36 \pm 0.14\,. \tag{55}$$

Our calculated result for ρ^2 is close to that in Eq. (54) but our form of $\xi(w)$ has significant curvature and deviates quickly from the linear fit. The curvature is, in fact, very well matched to that of the fit in Eq. (55), however, the value of ρ^2 listed in that case is very different to our calculated value.

In Ref. [15] we fitted E and Λ to the nonlinear form in Eq. (55) and fitted it exactly. We believe that part of the discrepancy observed here is due to our neglect of $1/\hat{M}_c$-corrections in the calculation of $\xi(w)$, the magnitude of which is exposed because of our newfound ability

[5]In our framework the minimum possible value for ρ^2 is 1/3 [15].

to constrain our parameters without referring to D-meson observables. Nevertheless, the agreement between this calculation and the data is reasonable, with the difference largest at $\omega_{\rm max}$ where it is a little more than one standard deviation. Hence $1/\hat{M}_c$-corrections cannot be too large.

In Fig. 3 we present our calculated form of $f_+^{DK}(t)$. The t-dependence is well-approximated by a monopole fit. Our favoured, calculated value of $f_+^{DK}(0) = 0.62$ is approximately 15% less than the experimental value [1]. We interpret this as a *gauge* of the size of $1/\hat{M}_c$-corrections. These corrections are expected to reduce the value of the D-meson leptonic decay constants from that obtained using Eq. (50). A 15% reduction in the D-meson leptonic decay constants in column one of Table I yields $f_D = 0.24\,{\rm GeV}$ and $f_{D_s} = 0.26\,{\rm GeV}$, values which are consistent with lattice estimates [24] and the latter with experiment [30].

We have also calculated $f_+^{D\pi}(t)$ and find that on the kinematically accessible domain, $0 < t < (m_D - m_\pi)^2$, the following monopole form provides an excellent *interpolation*

$$f_+^{D\pi}(t) = \frac{0.716}{1 - t/m_{\rm mon}^2}\,,\; m_{\rm mon} = 2.15\,{\rm GeV}\,. \tag{56}$$

We note that a naive vector meson dominance assumption would lead one to expect $m_{\rm mon} \approx m_{D^*} = 2.0\,{\rm GeV}$. Using (E,Λ) from Table I we obtain

$$R_\pi := \frac{Br(D\to \pi\ell\nu)}{Br(D\to K\ell\nu)} = 2.47\left|\frac{V_{cd}}{V_{cs}}\right|^2 = 0.13\,, \tag{57}$$

for $|V_{cd}/V_{cs}|^2 = 0.051 \pm 0.002$ [1], and in this ratio the bulk of the $1/\hat{M}_c$-corrections should cancel. Experimentally

$$R_\pi = \frac{Br(D^0\to \pi^- e^+\nu_e)}{Br(D^0\to K^- e^+\nu_e)} = 0.11^{+0.06}_{-0.03} \pm 0.1 \quad [1,38]\,, \tag{58}$$

$$R_\pi = 2\frac{Br(D^+\to \pi^0 e^+\nu_e)}{Br(D^+\to \bar{K}^0 e^+\nu_e)} = 0.17 \pm 0.05 \pm 0.03 \quad [39]\,. \tag{59}$$

We observe that if one makes the assumption of single-pole, D^* and D_s^* vector meson dominance for the t-dependence of the form factors $f_+^{D\pi}$ and f_+^{DK}, respectively, one obtains the simple formula

$$R_\pi = 1.97\left|\frac{f_+^{D\pi}(0)}{f_+^{DK}(0)}\right|^2\left|\frac{V_{cd}}{V_{cs}}\right|^2\,. \tag{60}$$

This approach has been employed [1] in order to estimate $f_+^{D\pi}(0)/f_+^{DK}(0) = 1.0^{+0.3}_{-0.2} \pm 0.04$ or $1.3 \pm 0.2 \pm 0.1$ from Eqs. (58) and (59). We calculate

$$\frac{f_+^{D\pi}(0)}{f_+^{DK}(0)} = 1.16\,. \tag{61}$$

It is incumbent upon us now to stress that we explicitly *do not* assume vector meson dominance. Our calculated results reflect only the importance and influence of the dressed-quark and -gluon substructure of the heavy mesons. This substructure is manifest in the

dressed propagators and bound state amplitudes, which fully determine the value of every quantity calculated herein. Explicit vector meson contributions would appear as pole terms in $\mathcal{V}_\mu^{f_1 f_2}(k_1,k_2)$, which are excluded in our Ansatz, Eq. (36). That simple-pole Ansätze provide efficacious interpolations of our results on the accessible kinematic domain is not surprising, given that the form factor must rise slowly away from its value at $t=0$ and the heavy meson mass provides a dominant intrinsic scale, which is modified slightly by the scale in the light-quark propagators and meson bound state amplitudes. Similar observations are true in the calculation of the pion form factor, as discussed in detail in Sec. 7.1 of Ref. [5] and Sec. 2.3.1 of Ref. [40].

In column two of Table I we present the results obtained when E and Λ are varied in order to obtain a best, weighted least-squares fit to: the lattice data on $f_+^{B\pi}$; the $B^0 \to \pi^- \ell^+ \nu$ branching ratio; and the experimental data on $\xi(w)$ reported in Ref. [28]. The latter introduce D-meson properties into our fitting constraints but their effect on our calculations is not very significant. The tabulated quantity most affected is the $B^0 \to \pi^- \ell^+ \nu$ branching ratio but this increases by only 15% and remains acceptably close to the experimental value. The effect that this modified fitting procedure has on the transition form factors is also small, as illustrated by the comparisons in Figs. 1-3. Not surprisingly, the largest effect is a uniform 5% increase in the magnitude of $f_+^{DK}(t)$.

In Table III we present the results obtained using the different functional form for the heavy-meson Bethe-Salpeter amplitude in Eq. (30). A direct comparison with the results in Table I indicates that our results are insensitive to such details and hence are robust. The binding energy, E, is unchanged and the width, $\tilde{\Lambda}$, is smaller, as expected since Eq. (30) does not decrease as rapidly with k^2 as the form in Eq. (29). A quantitative statement of this is that

$$\int_0^\infty dk^2 \, \left(\mathrm{e}^{-k^2/\Lambda^2}\right)^2 = \tfrac{1}{2}\,\Lambda^2\,, \tag{62}$$

$$\int_0^\infty dk^2 \, \left(\frac{\tilde{\Lambda}^2}{k^2+\tilde{\Lambda}^2}\right)^2 = \tilde{\Lambda}^2 \tag{63}$$

and $\tilde{\Lambda} = 0.92\,\mathrm{GeV} \sim \Lambda/\sqrt{2} = 1.0\,\mathrm{GeV}$ is just that reduction necessary to provide the same integrated strength for both amplitudes.

Tables IV and V provide a further elucidation of the impact of possible systematic errors in our calculation. These results are obtained through a repetition of the calculations that yield Table I but with f_B constrained to be 0.135 and 0.205 GeV, respectively, which are the outer limits estimated in an analysis of contemporary lattice simulations [24]. In the direct application of the methods of Ref. [17] to heavy mesons the value of f_B would be a prediction. Herein, since we do not calculate but instead fit the heavy-meson Bethe-Salpeter amplitude, f_B acts as a constraint on the width, Λ, of the Bethe-Salpeter amplitude, as seen in a comparison of Tables I, IV and V. The binding energy, E, is then the only true free parameter and it varies over a range of no more than 8%. Comparing these tables, we see that our results are not very sensitive to the value of f_B in the range we have explored; i.e., our results are robust.

We judge that the best description of the available data is obtained with $f_B = 0.17\,\mathrm{GeV}$, with a lower value, $f_B \to 0.135\,\mathrm{GeV}$, more acceptable than a higher one. The value of $E = 0.44\,\mathrm{GeV}$ that provides this best description can be compared with the value of $E_{\mathrm{bind}} \sim$

0.25-0.35 GeV obtained in a lattice NRQCD simulation [41]. The value of $\tilde{\Lambda} = 0.92$ GeV indicates that the heavy meson occupies a spacetime volume only 15% of that occupied by the pion.

IV. CONCLUSIONS

Using the same phenomenological Dyson-Schwinger equation (DSE) framework employed in successful studies of light meson observables as diverse as π-π scattering [42] and diffractive electroproduction of vector mesons [13], we have analysed semileptonic heavy $\rightarrow$ heavy and heavy $\rightarrow$ light meson transition form factors. In this application we introduced and explored a heavy-quark limit of the DSEs based on the observation that the mass function of heavy quarks evolves slowly with momentum.

With two parameters: E, the difference between the heavy-meson mass and the effective-mass of the heavy quark; and Λ, the width of the heavy meson Bethe-Salpeter amplitude, we obtained a uniformly good, robust description of all available $B \rightarrow \pi$ data with a prediction for $f_+^{B\pi}(t)$ on the kinematically accessible t-domain. In analysing $B \rightarrow D$, $D \rightarrow K$ and $D \rightarrow \pi$ transitions we estimated that $1/m_c$-corrections to our heavy-quark limit contribute no more than 15%. A significant feature of our study is the correlation of heavy $\rightarrow$ heavy and heavy $\rightarrow$ light transitions *and* their correlation with light meson observables, which are dominated by effects such as dynamical chiral symmetry breaking and confinement.

This study can be extended, with the application of the framework to semileptonic decays with vector meson final states using no additional parameters. It can also be improved, for example, by an exploration of the effect of more sophisticated Ansätze for the dressed-quark-W-boson vertex and of the inclusion of all amplitudes in the light-meson Bethe-Salpeter amplitude with refitted light-quark propagators.

A more significant qualitative improvement is the direct study of the Bethe-Salpeter equation for heavy mesons using the methods of Ref. [17]; Ref. [14] is the pilot. This programme involves the important step of critically analysing the reliability for heavy quarks of ladder-like truncations of the dressed-quark-antiquark scattering kernel in both the quark DSE and meson Bethe-Salpeter equation. Addressing this question and developing an efficacious truncation will allow a *correlation* of heavy- and light-meson observables via the few parameters that characterise the behaviour of the quark-quark interaction in the non-perturbative domain; i.e., relate both heavy- and light-meson observables to the long-range part of the quark-quark interaction.

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APPENDIX: A DERIVATION

A typical integral arising in the detailed analysis of Eq. (12) has the form

$$J = \int \frac{d^4k}{\pi^2}\,\frac{1}{k\cdot v - E}\,Z(k^2)\,\sigma([k-p_2]^2)\,. \tag{A1}$$

To simplify it we introduce a Laplace transform for the functions $Z(k^2)$ and $\sigma([k-p_2]^2)$:

$$Z(k^2) = \int_0^\infty ds\,\tilde{Z}(s)\mathrm{e}^{-sk^2}\,, \tag{A2}$$

$$\sigma([k-p_2]^2) = \int_0^\infty du\,\tilde{\sigma}(u)\mathrm{e}^{-u[k-p_2]^2} \tag{A3}$$

and a Gaussian representation of the heavy-quark propagator:

$$\frac{1}{k\cdot v - E} = \int_0^\infty d\alpha\,\mathrm{e}^{-\alpha(k\cdot v - E)}\,. \tag{A4}$$

Inserting these identities we obtain

$$J = \int\limits_0^\infty ds\,\tilde{Z}(s)\int\limits_0^\infty du\,\tilde{\sigma}(u)\int\limits_0^\infty d\alpha\,\int\frac{d^4k}{\pi^2}\,\exp\{-sk^2 - \alpha(k\cdot v - E) - u[k+p_2]^2\} \tag{A5}$$

$$= \int\limits_0^\infty ds\,\tilde{Z}(s)\int\limits_0^\infty du\,\tilde{\sigma}(u)\int\limits_0^\infty d\alpha\,\exp\left\{\alpha E - up_2^2 + (up_2 + \tfrac{1}{2}\alpha v)^2/(s+u)\right\} \tag{A6}$$

$$\int\frac{d^4k}{\pi^2}\,\exp\left\{-(s+u)\left[k + (up_2+\tfrac{1}{2}\alpha v)/(s+u)\right]^2\right\}\,.$$

Shifting variables: $k \to k - (up_2 + \frac{1}{2}\alpha v)/(s+u)$ and subsequently $\alpha \to (s+u)\alpha$, yields

$$J = \int\limits_0^\infty ds\,\tilde{Z}(s)\int\limits_0^\infty du\,\tilde{\sigma}(u) \tag{A7}$$

$$\int\limits_0^\infty d\alpha\,(s+u)\,\exp\left\{-(s+u)(\tfrac{1}{4}\alpha^2 - \alpha E) + u\alpha X - \frac{su}{s+u}\,p_2^2\right\}\int\frac{d^4k}{\pi^2}\,\mathrm{e}^{-(s+u)k^2}$$

$$= \int\limits_0^\infty ds\,\tilde{Z}(s)\int\limits_0^\infty du\,\tilde{\sigma}(u)\int\limits_0^\infty d\alpha\,\frac{1}{s+u}\,\exp\left\{-(s+u)(\tfrac{1}{4}\alpha^2 - \alpha E) - u\alpha X - \frac{su}{s+u}\,p_2^2\right\} \tag{A8}$$

where $X := -v\cdot p_2$, Eq. (46). Making use of the identities

$$\exp\left\{-\frac{su}{s+u}\,p_2^2\right\} = \sqrt{\frac{s+u}{\pi}}\,\int_{-\infty}^\infty d\tau\,\exp\left\{-s\tau^2 - u\left(\tau + \sqrt{p_2^2}\right)^2\right\}\,, \tag{A9}$$

$$\frac{1}{\sqrt{s+u}} = \frac{1}{\sqrt{\pi}}\,\int_{-\infty}^\infty dt\,\mathrm{e}^{-(s+u)t^2} \tag{A10}$$

we obtain

$$J = \frac{2}{\pi} \int_0^\infty d\alpha \int_{-\infty}^\infty d\tau \tag{A11}$$
$$\int_{-\infty}^\infty dt\, Z(\alpha^2 - 2\alpha E + \tau^2 + t^2)\, \sigma\left(\alpha^2 - 2\alpha E + 2\alpha X + (\tau + \sqrt{p_2})^2 + t^2\right) .$$

Introducing spherical polar coordinates

$$\alpha = u\,\nu\,, \tag{A12}$$
$$\tau = u\,\sqrt{1-\nu^2}\gamma\,, \tag{A13}$$
$$t = u\,\sqrt{1-\nu^2}\,\sqrt{1-\gamma^2}\,, \tag{A14}$$

with $u \in [0, \infty)$, $\nu \in [0, 1]$ and $\gamma \in [-1, 1]$, we arrive at

$$J = \frac{4}{\pi} \int_{-1}^1 \frac{d\gamma}{\sqrt{1-\gamma^2}} \int_0^1 d\nu \int_0^\infty du\, u^2\, Z(z_1)\, \sigma(z_2)\,, \tag{A15}$$

where, using $p_2^2 = -m_{H_2}^2$,

$$z_1 = u^2 - 2\,u\,\nu\,E\,, \tag{A16}$$
$$z_2 = u^2 - 2\,u\,\nu\,(E - X)\, - m_{H_2}^2 + 2\,i\,m_{H_2}\,u\,\sqrt{1-\nu^2}\,\sqrt{1-\gamma^2}\,. \tag{A17}$$

This is recognisably of the form in Eq. (41).

Structures more complicated than Eq. (A1) arise in deriving the complete form of $F_{q'}$, however, they can all be analysed and simplified using analogues of the method illustrated above.

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FIGURES

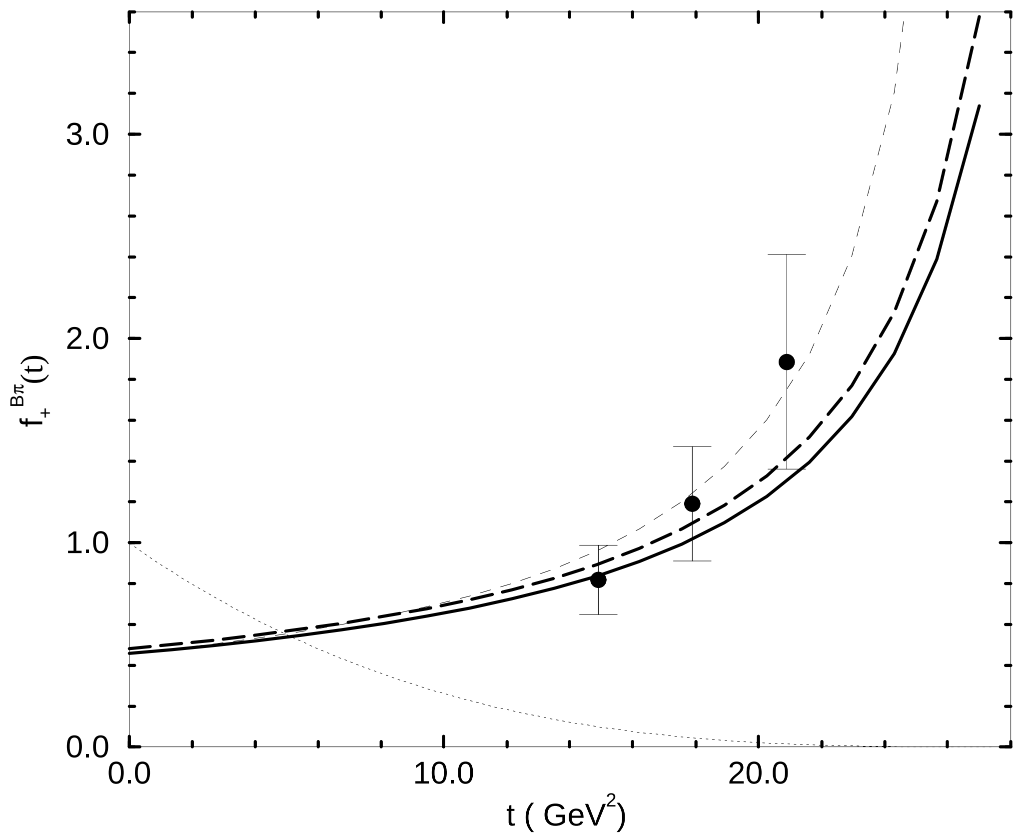

FIG. 1. Our calculated form of $f_+^{B\pi}(t)$: solid line - column 1, Table I; dashed line - column 2, Table I. For comparison, the data are the results of a lattice simulation [25] and the light, short-dashed line is a vector dominance, monopole model: $f_+(t) = 0.46/(1 - t/m_{B^*}^2)$, $m_{B^*} = 5.325\,\mathrm{GeV}$. The light, dotted line is the phase space factor $|f_+^{B\pi}(0)|^2\,[(t_+ - t)(t_- - t)]^{3/2}/(\pi m_B)^3$ in Eq. (11), which illustrates that the $B \to \pi e\nu$ branching ratio is determined primarily by the small-q^2 behaviour of this form factor.

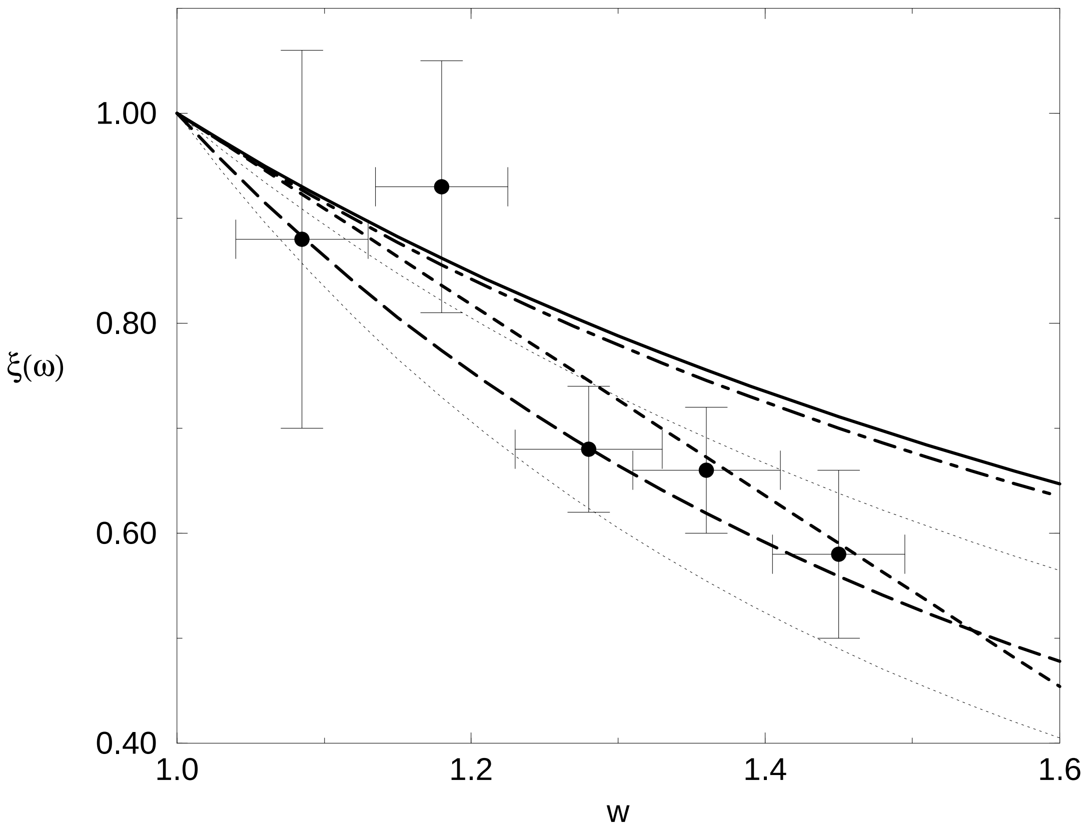

FIG. 2. A comparison of our calculated form of $\xi(w)$ with recent experimental analyses. Our results: solid line - column 1, Table I; dot-dashed line - column 2, Table I. Experiment: data points - Ref. [28]; short-dashed line - linear fit from Ref. [29], see our Eq. (54); long-dashed line - nonlinear fit from Ref. [29], see our Eq. (55). The two light, dotted lines are this nonlinear fit evaluated with the extreme values of ρ^2: upper line, $\rho^2 = 1.17$ and lower line, $\rho^2 = 1.89$.

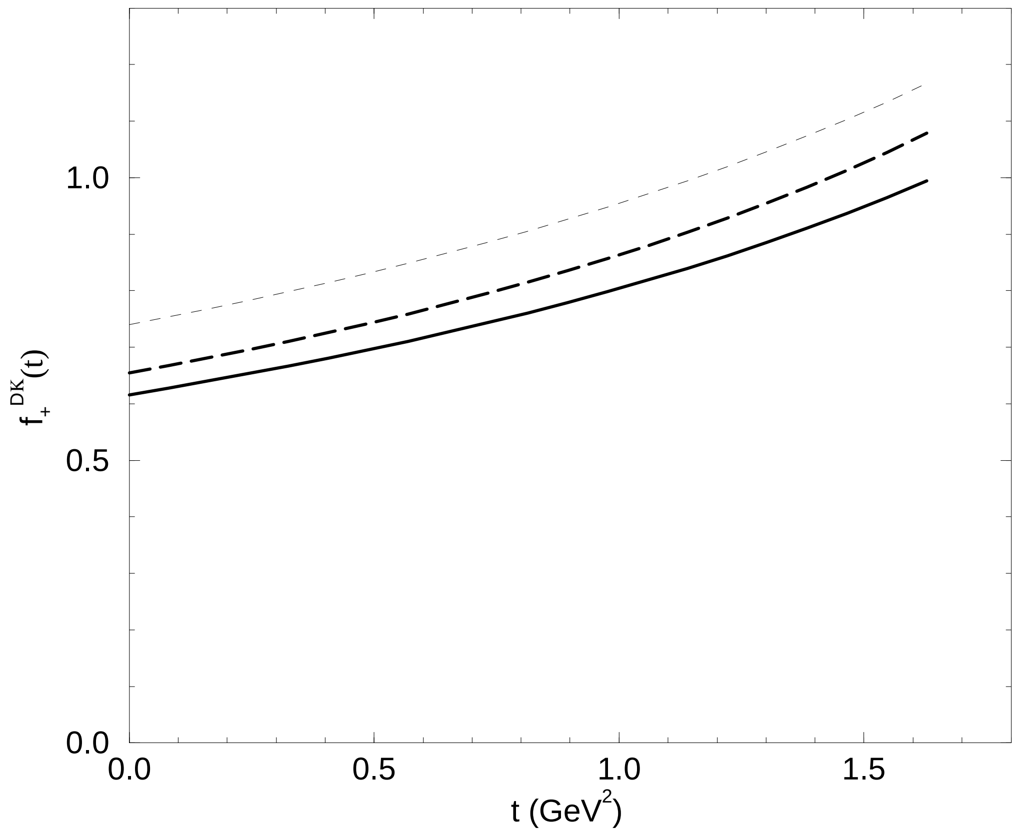

FIG. 3. Our calculated form of $f_+^{DK}(q^2)$: solid line - column 1, Table I; dashed line - column 2, Table I. For comparison the light, short-dashed line is a vector dominance, monopole model: $f_+(q^2) = 0.74/(1 - q^2/m^2_{D_s^*})$, $m_{D_s^*} = 2.11\,\mathrm{GeV}$.

TABLES

	DATA/ESTIMATES	$f_B = 0.170\,\mathrm{GeV}$	
(E,Λ) (GeV)		(0.442,1.408)	(0.465,1.405)
Σ^2/N		0.48	1.22
$f_+^{B\pi}(14.9\,\mathrm{GeV}^2)$	0.82 ± 0.17 [25]	$0.84^\dagger$	$0.89^\dagger$
$f_+^{B\pi}(17.9\,\mathrm{GeV}^2)$	1.19 ± 0.28 [25]	$1.02^\dagger$	$1.09^\dagger$
$f_+^{B\pi}(20.9\,\mathrm{GeV}^2)$	1.89 ± 0.53 [25]	$1.30^\dagger$	$1.41^\dagger$
$\mathrm{Br}(B^0 \to \pi^- \ell^+ \nu)$	$[1.8 \pm 0.4 \pm 0.3 \pm 0.2] \times 10^{-4}$ [26]	$2.0\ \times 10^{-4\dagger}$	$2.3\times 10^{-4\dagger}$
$f_+^{B\pi}(0)$	$0.18 \to 0.49$ [27]	0.46	0.48
$f_+^{DK}(0)$	$0.74\ \pm 0.03$ [1]	0.62	0.65
$\xi(1.085 \pm 0.045)$	0.88 ± 0.18 [28]	0.93	$0.93^\dagger$
$\xi(1.18 \pm 0.045)$	0.93 ± 0.12 [28]	0.86	$0.86^\dagger$
$\xi(1.28 \pm 0.050)$	0.68 ± 0.06 [28]	0.80	$0.79^\dagger$
$\xi(1.36 \pm 0.050)$	0.66 ± 0.06 [28]	0.76	$0.75^\dagger$
$\xi(1.45 \pm 0.045)$	0.58 ± 0.08 [28]	0.71	$0.70^\dagger$
ρ^2	$0.91 \pm 0.15 \pm 0.06$ $1.53 \pm 0.36 \pm 0.14$ [29]	0.87	0.92
f_{B_s} (GeV)	0.195 ± 0.035 [24]	0.184	0.184
f_{B_s}/f_B	1.14 ± 0.08 [24]	1.083	1.082
f_D (GeV)	0.200 ± 0.030 [24]	0.285	0.285
f_{D_s} (GeV)	0.220 ± 0.030 [24]	0.304	0.304
f_{D_s}/f_D	1.10 ± 0.06 [24]	1.066	1.066

TABLE I. A comparison of our calculated results with available data when we require $f_B = 0.170\,\mathrm{GeV}$, which is the central value estimated in Ref. [24], and use Eq. (29). In each column the quantities marked by † are those used to constrain the parameters (E,Λ) by minimising $\Sigma^2 := \sum_{i=1}^{N} ([y_i^{\mathrm{calc}} - y_i^{\mathrm{data}}]/\sigma(y)_i^{\mathrm{data}})^2$, where N is the number of data items used. The results in the first column assume that heavy-quark symmetry is valid for the b-quark but do not rely on this being true for the c-quark. We note that: 1) our values of f_D and f_{D_s} are obtained via Eq. (51) from f_B and f_{B_s}, respectively, using $m_B = 5.27$, $m_{B_s} = 5.375$, $m_D = 1.87$ and $m_{D_s} = 1.97\,\mathrm{GeV}$; 2) the experimental determination of ρ^2 is sensitive to the form of the fitting function, e.g., see Ref. [29]; 3) an analysis of four experimental measurements of $D_s \to \mu\nu$ decays yields $f_{D_s} = 0.241 \pm 0.21 \pm 0.30\,\mathrm{GeV}$ [30].

Reference	$f_+^{B\pi}(0)$
Our Result	0.46
Dispersion relations [27]	$0.18 \to 0.49$
Quark Model [31]	0.33 ± 0.06
Quark Model [32]	0.21 ± 0.02
Quark Model [33]	0.29
Light-Cone Sum Rules [34]	0.29 direct 0.44 pole dominance
Quark Confinement Model [35]	0.6
Quark Confinement Model [36,37]	0.53

TABLE II. A comparison of our favoured, calculated result for $f_+^{B\pi}(0)$ with a representative but not exhaustive list of values obtained using other theoretical tools. More extensive and complementary lists are presented in Refs. [27,33,37].

	DATA/ESTIMATES	$f_B = 0.170\,\mathrm{GeV}$
(E,Λ) (GeV)		(0.455,0.918)
Σ^2/N		0.46
$f_+^{B\pi}(14.9\,\mathrm{GeV}^2)$	0.82 ± 0.17 [25]	$0.84^\dagger$
$f_+^{B\pi}(17.9\,\mathrm{GeV}^2)$	1.19 ± 0.28 [25]	$1.02^\dagger$
$f_+^{B\pi}(20.9\,\mathrm{GeV}^2)$	1.89 ± 0.53 [25]	$1.32^\dagger$
$\mathrm{Br}(B^0 \to \pi^- \ell^+ \nu)$	$[1.8 \pm 0.4 \pm 0.3 \pm 0.2] \times 10^{-4}$ [26]	$2.0\ \times 10^{-4\dagger}$
$f_+^{B\pi}(0)$	$0.18 \to 0.49$ [27]	0.45
$f_+^{DK}(0)$	$0.74\ \pm 0.03$ [1]	0.62
$\xi(1.085 \pm 0.045)$	0.88 ± 0.18 [28]	0.92
$\xi(1.18 \pm 0.045)$	0.93 ± 0.12 [28]	0.84
$\xi(1.28 \pm 0.050)$	0.68 ± 0.06 [28]	0.77
$\xi(1.36 \pm 0.050)$	0.66 ± 0.06 [28]	0.72
$\xi(1.45 \pm 0.045)$	0.58 ± 0.08 [28]	0.67
ρ^2	$0.91 \pm 0.15 \pm 0.06$ $1.53 \pm 0.36 \pm 0.14$ [29]	1.03
f_{B_s} (GeV)	0.195 ± 0.035 [24]	0.180
f_{B_s}/f_B	1.14 ± 0.08 [24]	1.061
f_D (GeV)	0.200 ± 0.030 [24]	0.285
f_{D_s} (GeV)	0.220 ± 0.030 [24]	0.298
f_{D_s}/f_D	1.10 ± 0.06 [24]	1.044

TABLE III. A comparison of our calculated results with available data when we require $f_B = 0.170\,\mathrm{GeV}$, which is the central value estimated in Ref. [24], and use Eq. (30) . (See Table. I for additional remarks and an explanation of the symbols.)

	DATA/ESTIMATES	$f_B = 0.135\,\mathrm{GeV}$	
(E,Λ) (GeV)		(0.457,1.138)	(0.466,1.135)
Σ^2/N		0.50	0.97
$f_+^{B\pi}(14.9\,\mathrm{GeV}^2)$	0.82 ± 0.17 [25]	$0.86^\dagger$	$0.88^\dagger$
$f_+^{B\pi}(17.9\,\mathrm{GeV}^2)$	1.19 ± 0.28 [25]	$1.05^\dagger$	$1.08^\dagger$
$f_+^{B\pi}(20.9\,\mathrm{GeV}^2)$	1.89 ± 0.53 [25]	$1.36^\dagger$	$1.40^\dagger$
$\mathrm{Br}(B^0 \to \pi^- \ell^+ \nu)$	$[1.8 \pm 0.4 \pm 0.3 \pm 0.2] \times 10^{-4}$ [26]	$2.1 \times 10^{-4\dagger}$	$2.2 \times 10^{-4\dagger}$
$f_+^{B\pi}(0)$	$0.18 \to 0.49$ [27]	0.46	0.47
$f_+^{DK}(0)$	0.74 ± 0.03 [1]	0.64	0.65
$\xi(1.085 \pm 0.045)$	0.88 ± 0.18 [28]	0.92	$0.92^\dagger$
$\xi(1.18 \pm 0.045)$	0.93 ± 0.12 [28]	0.85	$0.85^\dagger$
$\xi(1.28 \pm 0.050)$	0.68 ± 0.06 [28]	0.78	$0.78^\dagger$
$\xi(1.36 \pm 0.050)$	0.66 ± 0.06 [28]	0.74	$0.73^\dagger$
$\xi(1.45 \pm 0.045)$	0.58 ± 0.08 [28]	0.69	$0.69^\dagger$
ρ^2	$0.91 \pm 0.15 \pm 0.06$ $1.53 \pm 0.36 \pm 0.14$ [29]	0.96	0.98
f_{B_s} (GeV)	0.195 ± 0.035 [24]	0.148	0.148
f_{B_s}/f_B	1.14 ± 0.08 [24]	1.096	1.096
f_D (GeV)	0.200 ± 0.030 [24]	0.227	0.227
f_{D_s} (GeV)	0.220 ± 0.030 [24]	0.244	0.244
f_{D_s}/f_D	1.10 ± 0.06 [24]	1.079	1.078

TABLE IV. A comparison of our calculated results with available data when we require $f_B = 0.135\,\mathrm{GeV}$, which is the lower bound estimated in Ref. [24], and use Eq. (29). (See Table. I for additional remarks and an explanation of the symbols.)

	DATA/ESTIMATES	$f_B = 0.205\,\mathrm{GeV}$	
(E,Λ) (GeV)		(0.469,1.677)	(0.479,1.678)
Σ^2/N		0.83	1.45
$f_+^{B\pi}(14.9\,\mathrm{GeV}^2)$	0.82 ± 0.17 [25]	$0.91^\dagger$	$0.94^\dagger$
$f_+^{B\pi}(17.9\,\mathrm{GeV}^2)$	1.19 ± 0.28 [25]	$1.11^\dagger$	$1.15^\dagger$
$f_+^{B\pi}(20.9\,\mathrm{GeV}^2)$	1.89 ± 0.53 [25]	$1.43^\dagger$	$1.49^\dagger$
$\mathrm{Br}(B^0 \to \pi^- \ell^+ \nu)$	$[1.8 \pm 0.4 \pm 0.3 \pm 0.2] \times 10^{-4}$ [26]	$2.4\ \times 10^{-4\dagger}$	$2.5 \times 10^{-4\dagger}$
$f_+^{B\pi}(0)$	$0.18 \to 0.49$ [27]	0.49	0.50
$f_+^{DK}(0)$	0.74 ± 0.03 [1]	0.66	0.68
$\xi(1.085 \pm 0.045)$	0.88 ± 0.18 [28]	0.93	$0.93^\dagger$
$\xi(1.18 \pm 0.045)$	0.93 ± 0.12 [28]	0.86	$0.86^\dagger$
$\xi(1.28 \pm 0.050)$	0.68 ± 0.06 [28]	0.80	$0.79^\dagger$
$\xi(1.36 \pm 0.050)$	0.66 ± 0.06 [28]	0.75	$0.75^\dagger$
$\xi(1.45 \pm 0.045)$	0.58 ± 0.08 [28]	0.71	$0.70^\dagger$
ρ^2	$0.91 \pm 0.15 \pm 0.06$ $1.53 \pm 0.36 \pm 0.14$ [29]	0.89	0.91
f_{B_s} (GeV)	0.195 ± 0.035 [24]	0.220	0.220
f_{B_s}/f_B	1.14 ± 0.08 [24]	1.071	1.071
f_D (GeV)	0.200 ± 0.030 [24]	0.344	0.344
f_{D_s} (GeV)	0.220 ± 0.030 [24]	0.363	0.363
f_{D_s}/f_D	1.10 ± 0.06 [24]	1.054	1.054

TABLE V. A comparison of our calculated results with available data when we require $f_B = 0.205\,\mathrm{GeV}$, which is the upper bound estimated in Ref. [24], and use Eq. (29). (See Table. I for additional remarks and an explanation of the symbols.)